Surrogate and reduced order modelling in stochastic dynamic micromagnetics

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Collaborators:

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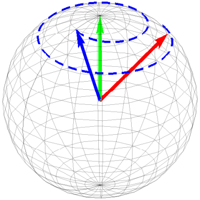
14th International symposium on Hysteresis Modeling and Micromagnetics (HMM 2025)

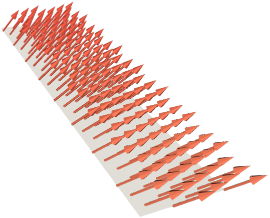
Stochastic Landau–Lifshitz–Gilbert (SLLG)

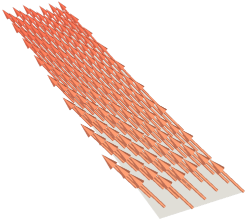
Find random magnetization $\boldsymbol{M} : \Omega \times [0,T] \times D \to \mathbb{S}^2$ such that

$$\mathrm{d}\boldsymbol{M} = [\boldsymbol{M} \times \boldsymbol{H}_{\text{eff}}(\boldsymbol{M}) - \boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{H}_{\text{eff}}(\boldsymbol{M}))]\, \mathrm{d}t + [\boldsymbol{M} \times \boldsymbol{g}] \circ \mathrm{d}W$$

Example: $\boldsymbol{H}_{\text{eff}}(\boldsymbol{M}) = \Delta \boldsymbol{M} + (0,0,-1)$







Reduction to a parametric LLG Equation

SLLG: Find $\boldsymbol{M} = \boldsymbol{M}(\omega, t, \boldsymbol{x}) : \Omega \times [0, T] \times D \to \mathbb{S}^2$

$$\mathrm{d}\boldsymbol{M} = [\boldsymbol{M} \times \Delta\boldsymbol{M} - \boldsymbol{M} \times (\boldsymbol{M} \times \Delta\boldsymbol{M})]\,\mathrm{d}t + [\boldsymbol{M} \times \boldsymbol{g}] \circ \mathrm{d}W$$

1. **Doss-Sussmann transform**

$$\boldsymbol{m}(\omega, t, \boldsymbol{x}) = e^{-W(\omega,t)(\cdot \times \boldsymbol{g})}\boldsymbol{M}(\omega, t, \boldsymbol{x})$$

$\boldsymbol{m}$ solves a random coefficient PDE [Goldys, Le, Tran 2016, *J. Diff. Eq.*]

2. **Lévy-Ciesielski parametrization**

$$W(\boldsymbol{y}, t) = \sum_{n \in \mathbb{N}} y_n \eta_n(t) \qquad \boldsymbol{y} \in \mathbb{R}^{\mathbb{N}} \text{ parameters}$$

Parametric LLG: Find $\boldsymbol{m} = \boldsymbol{m}(\boldsymbol{y}, t, \boldsymbol{x}) : \mathbb{R}^{\mathbb{N}} \times [0, T] \times D \to \mathbb{S}^2$

$$\partial_t \boldsymbol{m} = \boldsymbol{m} \times \Delta\boldsymbol{m} - \boldsymbol{m} \times (\boldsymbol{m} \times \Delta\boldsymbol{m}) + F(W(\boldsymbol{y}), \boldsymbol{m})$$

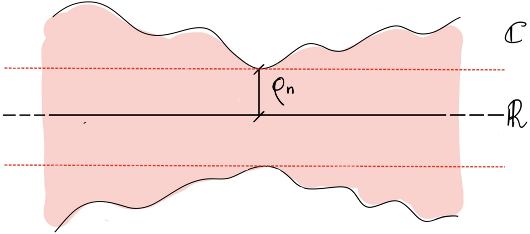
Parameter-to-magnetization map regularity

$$\boldsymbol{m} : \boldsymbol{y} \xrightarrow{\text{Levy-Ciesielski}} W(\boldsymbol{y}) \xrightarrow{\text{LLG}^{-1}} \boldsymbol{m}(\boldsymbol{y})$$

Parameter-to-magnetization map regularity [ADFST'25]

There exist $(\rho_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that
$\boldsymbol{m} : \mathbb{R}^{\mathbb{N}} \to \mathbb{M}$ extends as a **holomorphic**, **bounded** function to

$$\mathbb{R}^{\mathbb{N}} \subset \left\{(z_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\mathrm{Im}\{z_n\}| < \rho_n\right\} \subset \mathbb{C}^{\mathbb{N}}$$



An, Dick, Feischl, S, and Tran 2025, *JUQ*.

What are $(\rho_n)_n$ for SLLG?

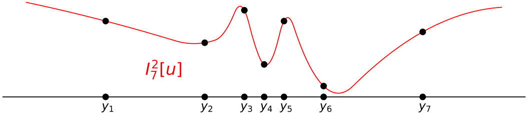
	Holder $0 < \alpha < 1$	**Lebesgue** $1 < q < \infty$
Coefficients $\mathbb{W}$	$C^{\alpha/2}([0,T])$	$L^q(0,T)$
Magnetizations $\mathbb{M}$	$C^{1+\alpha/2,2+\alpha}(D_T)^3$	$W^{1,q}(0,T;C^{2+\alpha}(D)^3)$
$\Longrightarrow (\rho_n)_n$	$\approx 2^{\frac{(1-\alpha)}{2}\lceil \log_2(n) \rceil}$	$\approx \begin{cases} 2^{\frac{3}{2}\lceil \log_2(n) \rceil} & \text{if } \nu_n = 1 \\ 2^{\frac{1}{2}\lceil \log_2(n) \rceil} & \text{if } \nu_n > 1 \end{cases}$

$$\|\partial^{\boldsymbol{\nu}} \boldsymbol{m}(\boldsymbol{y})\|_{\mathbb{M}} \lesssim \boldsymbol{\nu}! \left(\prod_{n \in \mathrm{supp}(\boldsymbol{\nu})} \rho_n{}^{-\nu_n} \right) \qquad \forall \boldsymbol{\nu} \in \mathbb{N}_0^s, \ \boldsymbol{y} \in \mathbb{R}^{\mathbb{N}}$$

Sparse Grid Interpolation of $\boldsymbol{m} : \boldsymbol{\Gamma} \to \mathbb{M}$

1 parameter: $\boldsymbol{\Gamma} = \mathbb{R}^1$

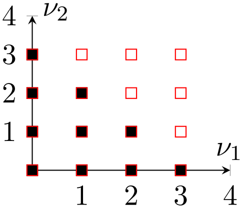
$$I_\nu \boldsymbol{m}(y) = \sum_{z=y_1,\dots,y_\nu} \boldsymbol{m}(z) L_z(y)$$



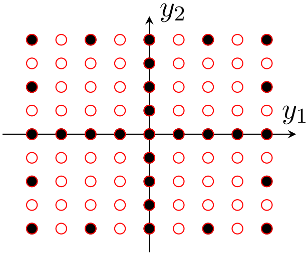
s parameters: $\boldsymbol{\Gamma} = \mathbb{R}^s$

$$\mathcal{I}_\Lambda \boldsymbol{m}(\boldsymbol{y}) = \sum_{\boldsymbol{\nu} \in \Lambda} \left(\bigotimes_{n=1}^{s} I_{\nu_n} - I_{\nu_n - 1} \right) \boldsymbol{m}(\boldsymbol{y})$$

$$= \sum_{\boldsymbol{z} \in \mathcal{H}_\Lambda} \boldsymbol{m}(\boldsymbol{z}) L_{\boldsymbol{z}}(\boldsymbol{y})$$



$\Lambda \subset \mathbb{N}_0^s$



$\mathcal{H}_\Lambda \subset \boldsymbol{\Gamma} = \mathbb{R}^s$

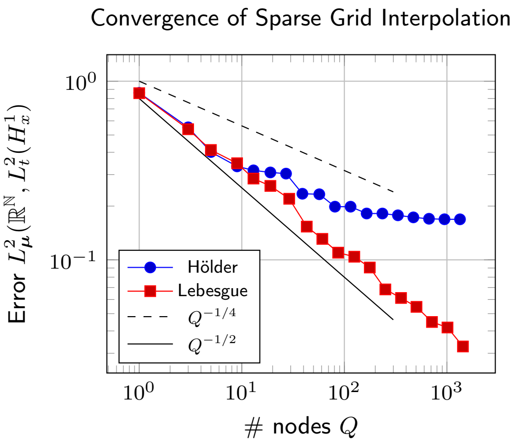
Convergence of Sparse Grid Interpolation

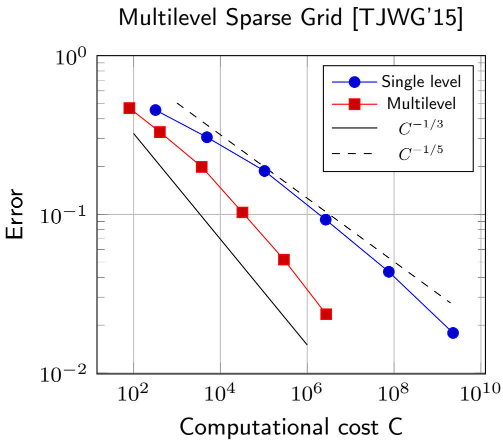
$$\exists \tau > 0 : \boxed{C_\tau} < \infty \xrightarrow{[BG`04],[NTT`16]} \|u - \mathcal{I}_\Lambda u\|_{L^2_\mu(\mathbb{R}^s)} \leq \boxed{C_\tau} \#\mathcal{H}_\Lambda^{-\tau}$$

	Holder $0 < \alpha < 1$	**Lebesgue** $1 < q < \infty$
$\Rightarrow (\rho_n)_n$	$\rho_n = 2^{\frac{(1-\alpha)}{2}\lceil \log_2(n) \rceil}$	$\begin{cases} 2^{\frac{3}{2}\lceil \log_2(n) \rceil} & \text{if } \nu_n = 1 \\ 2^{\frac{1}{2}\lceil \log_2(n) \rceil} & \text{if } \nu_n > 1 \end{cases}$
τ	$\frac{1}{2}$	$\frac{1}{2}$
$\boxed{C_\tau}$	$\approx e^{\sqrt{s}}$	s-**independent**

Nobile, Tamellini, and Tempone 2016, *Numer. Math.*
Bungartz and Griebel 2004, *Acta Numer.*

SG-SLLG numerics with SGMethods*





$*$ Python, `https://github.com/andreascaglioni/SGMethods`
Teckentrup, Jantsch, Webster, and Gunzburger 2015, *JUQ*.

POD-TPS Algorithm

Proper Orthogonal Decomposition–Tangent Plane Scheme

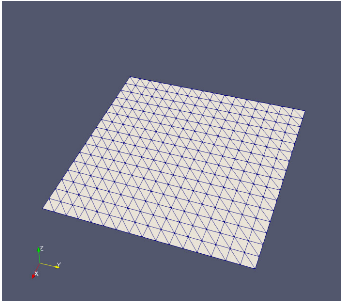
Goal: Obtain a **Reduced Basis** for the LLG velocities

Algorithm 1 Offline Phase (s, N_y, N_t, $\boldsymbol{m}^0$, λ)

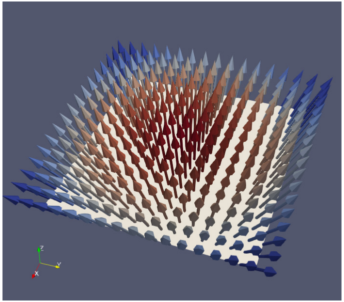
1: **for** $i = 1, \ldots, N_y$ **do**
2: Sample $\boldsymbol{y}_i \in \mathbb{R}^s$
3: $\boldsymbol{v}^1(\boldsymbol{y}_i), \ldots, \boldsymbol{v}^{N_t}(\boldsymbol{y}_i) \leftarrow \texttt{FEM-TPS}(N_t, \boldsymbol{m}^0, \lambda, \boldsymbol{y}_i)$ ▷ $\boldsymbol{v}^j(\boldsymbol{y}_i) \in \mathbb{R}^{N_h}$
4: **end for**
5: Assemble $S = \left[\boldsymbol{v}^1(\boldsymbol{y}_1) | \ldots | \boldsymbol{v}^{N_t}(\boldsymbol{y}_{N_y})\right]$ ▷ $S \in \mathbb{R}^{N_h \times (N_t \cdot N_y)}$
6: $U, \Sigma, V \leftarrow \texttt{SVD}(S)$ ▷ $S = U\Sigma V^T$
7: **return** Σ, U ▷ Columns U : Reduced Basis

Numerical examples: A perturbed relaxation problem

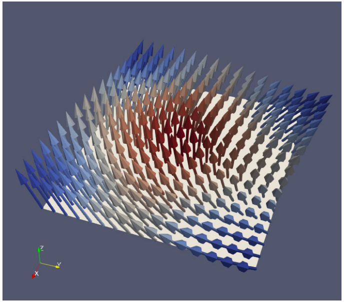
- T= 1
- $N_t = 10^3 + 1$ time steps
- $N_h = 512$ mesh elements
- 1D parameters: $\boldsymbol{y} = y \in \mathbb{R}$



Domain & mesh (snapshots)

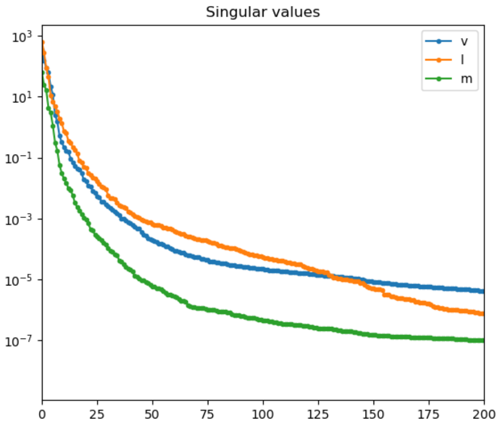


sample $\boldsymbol{m}(t = 0)$

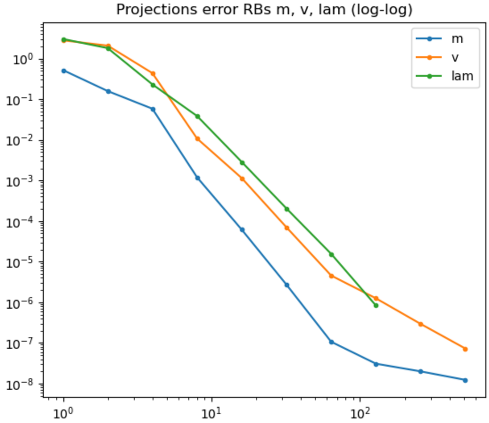


sample $\boldsymbol{m}(t = 1)$

Numerical examples: Offline Phase

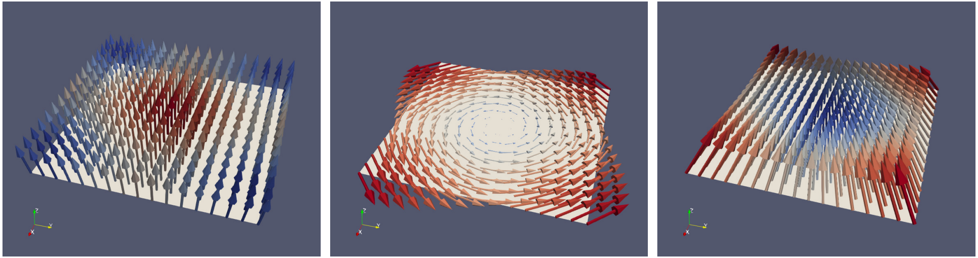


First 200 singular values



Projection error of new sample into reduced spaces

Numerical examples: Offline Phase



First 3 magnetization reduced basis function

Tangent Plane Scheme (TPS) Algorithm [AFKL'21]

- $\boldsymbol{V}$ (general) linear space of velocities : $D \to \mathbb{R}^3$
- Tangent Plane to $\boldsymbol{m}$: $\mathcal{K}(\boldsymbol{m}) = \left\{\boldsymbol{v} \in \boldsymbol{V} : \int_D (\boldsymbol{m} \cdot \boldsymbol{v})\boldsymbol{\phi} = 0\ \forall \boldsymbol{\phi} \in \boldsymbol{V}\right\}$

Algorithm 2 Online Phase: `TPS`$(N_t, \boldsymbol{V}, \boldsymbol{m}^0, \lambda, \boldsymbol{y})$

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for For $j = 0, 1, ..., N_t - 1$ **do** $\qquad \triangleright\ \tau = \frac{1}{N_t - 1}$

Find $\boldsymbol{v}^j \in \mathcal{K}(\boldsymbol{m}^j)$:

$$\lambda \boldsymbol{v}^j + \boldsymbol{m}^j \times \boldsymbol{v}^j - \Delta\left(\boldsymbol{m}^j + \tau \boldsymbol{v}^j\right) = F(W(\boldsymbol{y}, j\tau), \boldsymbol{m}^j) \qquad \text{in } \mathcal{K}(\boldsymbol{m}^j)'$$

$\boldsymbol{m}^{j+1} = \boldsymbol{m}^j + \tau \boldsymbol{v}^j$

end for

return $\boldsymbol{m}^j, \boldsymbol{v}^j$ for $j = 0, \dots, N_t$ $\qquad \triangleright\ \boldsymbol{m}^j \approx \boldsymbol{m}(\boldsymbol{y}, t_j),\ \boldsymbol{v}^j \approx \partial_t \boldsymbol{m}(\boldsymbol{y}, t_j)$

- `FEM-TPS`: $\boldsymbol{V}$ = FEM Space*
- `POD-TPS`: $\boldsymbol{V}$ = span(Reduced Basis)†

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* `stochllg` (Python, to be released) `https://github.com/andreascaglioni/stochllg`

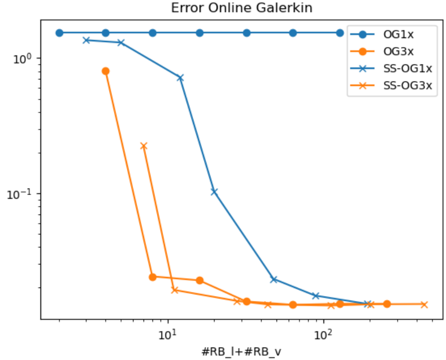
† `rom-sllg` (Python, to be released) `https://github.com/andreascaglioni/rom-sllg`

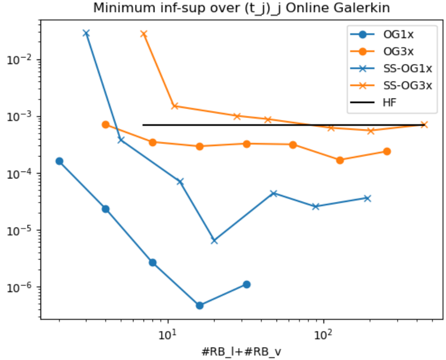
Akrivis, Feischl, Kovács, and Lubich 2021, *Math. Comp.*

Numerical examples: Online Phase

Convergence under reduced bases enlargement

- **OG-1x** ($\#\mathrm{RB}_\lambda = \#\mathrm{RB}_{\boldsymbol{v}}$)
- **OG-3x** ($\#\mathrm{RB}_{\boldsymbol{v}} = 3\#\mathrm{RB}_\lambda$)
- Supremizer-stabilized [GV'12]



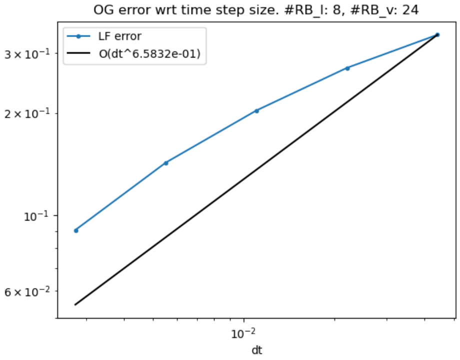


Gerner and Veroy 2012, *SIAM J. Sci. Comput.*

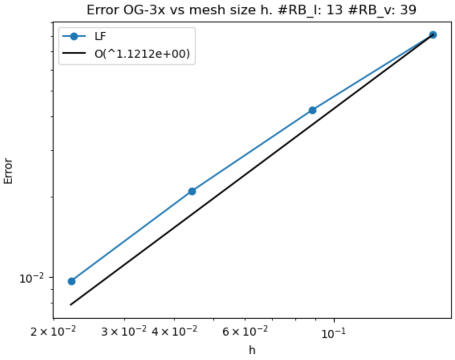
Numerical examples: Online Phase

t- and h-refinement

t-refinement (online)



h-refinement (snapshots)



Thanks for Listening!

Xin An, Josef Dick, Michael Feischl, AS, and Thanh Tran (2025).
Sparse grid approximation of nonlinear SPDEs: The Landau–Lifshitz–Gilbert equation.
In: JUQ 13 (2), pp. 472-517, DOI: 10.1137/24M1646054.

Fernando Henriquez, Michael Feischl, AS (2025+).
Reduced Order Modelling of the Stochastic Landau–Lifshitz–Gilbert equation.
In preparation.

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More topics on
Surrogate & Reduced Order Modelling of SLLG

- Why Uncertainty Quantification (UQ)?
- Stochastic PDEs with Gaussian noise
- General parameter-to-solution map regularity
- Sparse Grid Interpolation: (Hyper)parameter selection
- Numerical examples POD-TPS: Online Phase (h- and t-refinement)
- Comparison of SG-SLLG and POD-TPS
- Outlook & Open Problems

Why Uncertainty Quantification (UQ)?

	Deterministic PDE Numerics	**Uncertainty Quantification**
Problem	PDE $\mathcal{R}(a, u) = 0$	Random PDE $\mathcal{R}(a(\omega), u(\omega)) = 0$
Solution	$u : D \to \mathbb{R}$	$u : \Omega \times D \to \mathbb{R}$
Methods	Finite Differences FEM, DG, ...	Monte Carlo, **sparse grids**, **reduced bases**, ...

UQ relevant when:

- Unknown problem data (e.g. a)
- Measurement errors on problem data (e.g. a)
- Modelling of **noise in physics**, finance, biology, ...
- ...

Stochastic PDEs with Gaussian noise

$D \subset \mathbb{R}^3$, $T > 0$, $U^0 : D \to \mathbb{R}^3$, probability space $(\Omega, \mathcal{E}, \mathcal{P})$

$$\text{Find } U : \Omega \times D_T \to \mathbb{R}^3 \ : \ \text{a.e. in } \Omega$$

$$\begin{cases} \mathrm{d}U = \mathfrak{D}(U, \cdot, \cdot)\mathrm{d}t + \mathfrak{N}(U, \cdot) \circ \mathrm{d}W & \text{on } D_T := [0, T] \times D \\ \partial_n U = 0 & \text{on } [0, T] \times \partial D \\ U(\cdot, 0, \cdot) = U^0 & \text{on } D \end{cases}$$

Goal: Approximate $(\omega, t, \boldsymbol{x}) \mapsto U(\omega, t, \boldsymbol{x})$

Challenges:

- Not "just" a random coefficient PDE
- Curse of dimensionality
- Non-compact parameter space
- Nonlinear problem
- (Reduced sample path regularity)

General parameter-to-solution map regularity

Residual	$\mathcal{R} : \mathbb{W}_{\mathbb{R}} \times \mathbb{U}_{\mathbb{R}} \to R$	
Solution operator	$\mathcal{U} : \mathbb{W}_{\mathbb{R}} \to \mathbb{U}_{\mathbb{R}}$	$: \mathcal{R}(\cdot, \mathcal{U}(\cdot)) = 0$
Complex extension residual	$\mathcal{R} : \mathbb{W} \times \mathbb{U} \to R$	

Assumption 1: $\mathcal{U}$ uniformly bounded on $\mathbb{W}_{\mathbb{R}}$
Assumption 2: $\mathcal{R}$ Fréchet differentiable, $\partial_u \mathcal{R}$ homeomorphism
Assumption 3: $\partial_W \mathcal{R}(W, \mathcal{U}(W))$, $\partial_u \mathcal{R}(W, \mathcal{U}(W))$ depend continuously on $\mathcal{U}(W)$
Assumption 4: $\exists \varepsilon > 0, \boldsymbol{\rho} \subset \mathbb{R}_+^{\mathbb{N}}$ such that $W(\mathbf{B}_{\boldsymbol{\rho}}(\boldsymbol{y})) \subset B_\varepsilon(W(\boldsymbol{y}))$

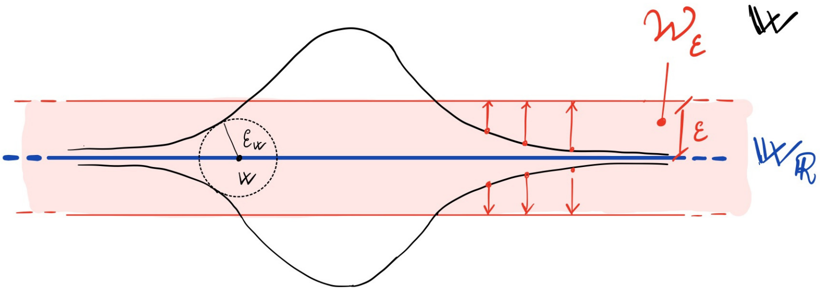
General parameter-to-solution map regularity [ADFST'25]

$$u : \boldsymbol{B}_{\boldsymbol{\rho}}(\boldsymbol{y}) \xrightarrow{\mathcal{U} \circ W} \mathbb{U} \qquad \forall \boldsymbol{y} \in \mathcal{X}_{\mathbb{R}},$$

is a **holomorphic, uniformly bounded** extension.

$$\Rightarrow \|\partial^{\boldsymbol{\nu}} u\|_{L^2_\mu(\mathbb{R}^{\mathbb{N}}, \mathbb{U})} \lesssim \boldsymbol{\nu}! \left(\prod_{n \in \operatorname{supp}(\boldsymbol{\nu})} \rho_n{}^{-\nu_n} \right) \qquad \forall \boldsymbol{\nu} \in \mathbb{N}_0^s$$

Solution operator regularity (Proof sketch)



$$\mathcal{W}_\varepsilon := \{W \in \mathbb{W} : \mathrm{dist}_\mathbb{W}(W, \mathbb{W}_\mathbb{R}) < \varepsilon\}, \;\; \varepsilon > 0$$

Tools:

- Fundamental theorem of calculus
- Nonlinear Gronwall inequality [Dragomir 2003, *Nova Science*]
- Implicit function theorem

Parameter-to-solution map regularity

Assumption 4: Sparsity For all $\varepsilon > 0$ the exists $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$ such that*

$$\boldsymbol{z} \in \mathsf{B}_{\boldsymbol{\rho}}(\boldsymbol{y}) \Rightarrow W(\boldsymbol{z}) = \sum_n z_n \eta_n \in B_\varepsilon(W(\boldsymbol{y}))$$

Parameter-to-solution map regularity [ADFST'25]

$$u : \boldsymbol{B}_{\boldsymbol{\rho}}(\boldsymbol{y}) \xrightarrow{\mathcal{U}\circ W} \mathbb{U} \qquad \forall \boldsymbol{y} \in \mathcal{X}_{\mathbb{R}},$$

is a **holomorphic, uniformly bounded** extension.

$$\Rightarrow \|\partial^{\boldsymbol{\nu}} u\|_{L^2_\mu(\mathbb{R}^{\mathbb{N}},\mathbb{U})} \lesssim \boldsymbol{\nu}! \left(\prod_{n\in\mathrm{supp}(\boldsymbol{\nu})} {\rho_n}^{-\nu_n} \right) \qquad \forall \boldsymbol{\nu} \in \mathbb{N}_0^s$$

* $\boldsymbol{B}_{\boldsymbol{\rho}}(\boldsymbol{y}) := \bigotimes_{n\in\mathbb{N}} B_{\rho_n}(y_n)$, where $B_{\rho_n}(y_n) := \{z \in \mathbb{C} : |z - y_n| < \rho_n\} \in \mathbb{C}$

Sparse Grid Interpolation: 2 Problems

$$\mathcal{I}_\Lambda \boldsymbol{m}(\boldsymbol{y}) = \sum_{\boldsymbol{z} \in \mathcal{H}_{\boxed{\Lambda}}} \boxed{\boldsymbol{m}(\boldsymbol{z})}\, L_{\boldsymbol{z}}(\boldsymbol{y})$$

- Sample $\boxed{\boldsymbol{m}(\boldsymbol{z})} \in \mathbb{M} \to$ **Tangent Plane Scheme** (TPS)
- Define $\boxed{\Lambda} \subset \mathbb{N}_0^s$ $(s \gg 1) \to$ **Profit-based selection** [BG'04]

$$\mathcal{P}_{\boldsymbol{\nu}} = \frac{\text{Value of } \boldsymbol{\nu}}{\text{Work of } \boldsymbol{\nu}}$$

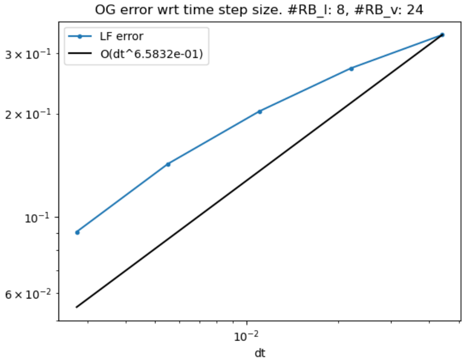
regularity analysis $\boldsymbol{y} \mapsto \boldsymbol{m}(\boldsymbol{y})$

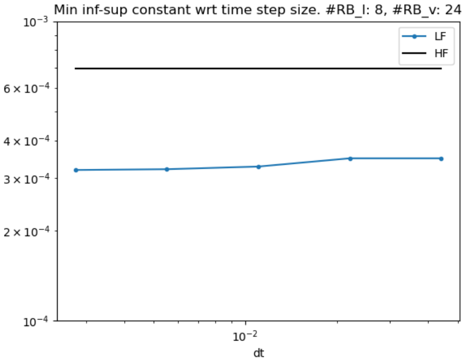
$$\exists \tau \in (0,1) : C_\tau^\tau = \sum_{\boldsymbol{\nu}} \mathcal{P}_{\boldsymbol{\nu}}^\tau w_{\boldsymbol{\nu}} < \infty \xrightarrow{[NTT'16]} \|\boldsymbol{m} - \mathcal{I}_\Lambda \boldsymbol{m}\|_{L^2_\mu(\mathbb{R}^{\mathbb{N}})} \leq C_\tau \# \mathcal{H}_\Lambda^{1-\frac{1}{\tau}}$$

Bungartz and Griebel 2004, *Acta Numer.*
Nobile, Tamellini, and Tempone 2016, *Numer. Math.*

Numerical examples: Online Phase

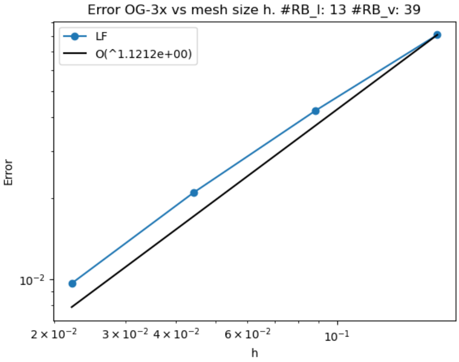
Convergence under t-refinement **online**

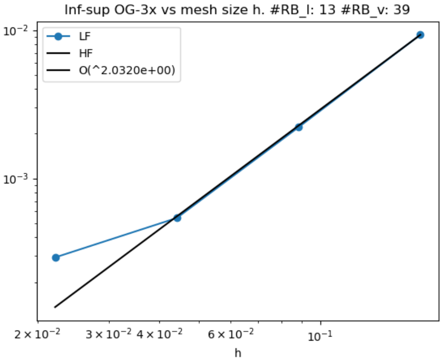




Numerical examples: Online Phase

Convergence under h-refinement of **snapshots**





Comparison of SG-SLLG and POD-TPS

	Sparse Grid (SG)-SLLG	**POD-TPS**
Approach	Non-intrusive	Intrusive
Offline	FEM on *specific* collocation nodes	FEM on “free” samples $\to$ POD
Online	Interpolate collocation samples	TPS on reduced space
Advantages	Simplicity, clearer relation between regularity and convergence	Possibly effective with fewer method’s parameters to select
Challenges	Selecting nodes and multi-index set	Complex implementation; indirect relation regularity–convergence
Convergence	Limited to rate $1/2$; Depends on holomorphic regularity $\boldsymbol{y} \mapsto \boldsymbol{m}(\boldsymbol{y})$	Depends on decay of Kolmogorov N-widths
Applications	General high-dimensional approximation problems	Limited to Galerkin-type problems

Outlook & Open Problems

SG-SLLG (Sparse Grid Interpolation)

- Can convergence rate $1/2$ be improved?
- A-posteriori estimation and adaptive Sparse Grid selection [FS'21]

POD-TPS (Reduced basis)

- Deduce Kolmogorov N-widths decay from holomorphic regularity(extend [CD'15])
- Selecting snapshot parameters $(\boldsymbol{y}_i)_{i=1}^{N_y}$: How many? What distribution?
- POD-TPS (Online Phase): How many reduced bases?
- Does POD-TPS beats SG-SLLG for some examples (energy conservation)?

Feischl and S 2021, *Comp. Math. Appl.*
Cohen and DeVore 2015, *IMA J. Numer. Anal.*